Industry Equilibrium with Open-Source and Proprietary Firms

Gastón Llanes[a,*], Ramiro de Elejalde[b]

[a]*Escuela de Administración, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile*
[b]*Facultad de Economía y Negocios, ILADES - Universidad Alberto Hurtado, Erasmo Escala 1835, Santiago, Chile*

Abstract

We present a model of industry equilibrium to study the coexistence of open-source and proprietary firms. Two novel aspects of the model are (i) participation in open source arises as the optimal decision of profit-maximizing firms, and (ii) open-source and proprietary firms may (or may not) coexist in equilibrium. Firms decide their type and investment in R&D, and sell packages composed of a primary good and a complementary private good. Open-source firms share their technological advances on the primary good, whereas proprietary firms keep their innovations private. The main contribution of the paper is to determine conditions under which open-source and proprietary firms coexist in equilibrium. Interestingly, this equilibrium is characterized by an asymmetric market structure, with few large proprietary firms and many small open-source firms. We also study the limiting economy and present conditions under which large numbers favor cooperation in R&D.

Keywords: Industry Equilibrium, Coexistence, Open Source, Complementarity, Technology Sharing, Cooperation in R&D
JEL: O31, L17, D43

1. Introduction

Collaboration in research enhances the chances of discovery and creation, not only for scientific discoveries, but also for commercial innovations. However, innovators face incentives to limit competitors' access to their innovations. According to the traditional view in the economics of innovation, innovators innovate because doing so allows them to obtain a monopolistic advantage over their competitors. Therefore, innovators should prevent others from gaining access to their discoveries, either by keeping them secret or by protecting them with patents.

This view contrasts with the free/open-source development model, in which innovators voluntarily choose to disclose their technological improvements so that other

*Corresponding author. Tel: +56 2 354 4033. Fax: +56 2 553 1672.
Email addresses: gaston@llanes.com.ar (Gastón Llanes), ramirode@gmail.com (Ramiro de Elejalde)

innovators can copy, use, and improve them free of charge. But if everybody has access to the same technologies, how do developers benefit from collaboration? What do they receive in exchange for renouncing their monopolistic advantages?

The answer is that open-source developers may benefit from participating in open-source projects by selling goods and services that are complementary to the open-source good. For example, IBM announced in 2001 that it was going to invest over 1 billion dollars in Linux, and today provides support for over 500 software products running on Linux, and has more than 15,000 Linux-related customers worldwide.[1]

Still, questions remain regarding what determines the choice of development model for profit-maximizing firms, why open-source and proprietary firms coexist in the same markets, and the implications of coexistence on market structure and investment incentives. Existing literature has yet to address these questions, which are the main focus of this paper.

We present a model of industry equilibrium with endogenous technology sharing. Firms sell packages composed of a primary good, such as software, and a complementary good, such as a smartphone, tablet PC, or support and training services. Firms choose their development model (open-source or proprietary), how much to invest in product development, and the price of their products. Open-source firms share their improvements to the main product, whereas proprietary firms, develop their products independently of other firms. Consumers value the quality of both goods (vertical differentiation) but also have idiosyncratic tastes for the products of different firms (horizontal differentiation).

We find the equilibrium may have both types of firms or only open-source firms. In the equilibrium with coexistence, the market structure is asymmetric, with few large proprietary firms and many small open-source firms. This finding is consistent with the observations of recent surveys. Seppä (2006) compares both types of firms and finds open-source firms tend to be smaller than proprietary firms. Bonaccorsi and Rossi (2004) show the most important motive for firms to participate in open-source projects is that participation allows small firms to innovate.

The equilibrium depends on the resolution of a trade-off between free-riding and collaboration, which is governed by a parameter measuring the degree of public good of the investment in R&D (i.e., the effect of total vs. individual contributions on quality). When open-source firms invest in R&D, they increase quality for all firms in the project. As a consequence, open-source firms are able to appropriate a smaller fraction of their investment, in comparison with proprietary firms. Nevertheless, open-source firms share their advances on the primary good, which means that even though each firm may individually invest less than a proprietary firm, the total investment of all firms in the project may be larger than the investment of a proprietary firm.

When the degree of public good of the investment in R&D is high, free-riding is important, which leads to lower individual investments for open-source firms. As a consequence, proprietary firms have an advantage over open-source firms in terms of market

[1] See `http://www.ibm.com/linux/` (accessed May 15, 2012).

share and price. On the other hand, open-source firms benefit from lower development costs. Therefore, both types of firms coexist in equilibrium: some firms choose to be proprietary, have a high investment in R&D, and benefit from high market shares and prices, and other firms choose to be open source and benefit from lower development costs.

For intermediate degrees of public good of the investment in R&D, free-riding becomes less important, and the difference in investment between open-source and proprietary firms becomes smaller. If the market-share advantage of proprietary firms is not large enough to compensate for the higher development costs, all firms choose the open-source development model. Nevertheless, a proprietary firm would invest more and produce a higher-quality product than open-source firms, so open-source prevents the entry of a higher-quality product.

Finally, when the degree of public good of the investment in R&D is low, the positive effects of collaboration are stronger than the negative effects of free-riding, and open-source firms have higher (total) investment than proprietary firms (individual investments are similar, but open-source firms share their investments). In this case, all firms choose the open-source development model to benefit from higher market shares and lower development costs than proprietary firms.

In the market equilibrium, welfare is suboptimal because of the public-good problem in open source and the duplication of effort of proprietary firms. In section 5, we show that a subsidy to open-source development can improve welfare not only because it increases the investment in R&D, but also because it encourages commercial firms to participate in open source, thereby enhancing collaboration.

The baseline model assumes symmetric consumer preferences for open-source and proprietary products. However, given that open-source packages are based on the same primary good, open-source products are likely to be more similar than the products of proprietary firms. In section 6, we modify the baseline model to allow for a higher cross-price elasticity between open-source products. We find the main result of the paper still holds: when open-source and proprietary firms coexist, the market share of proprietary firms is higher than that of open-source firms. However, in this case, we also find that if the substitution between open-source products is large enough, equilibria exist in which all firms choose the proprietary development model.

We also study investment incentives and market structure under free entry. When entry costs are small, the number of firms is large and the market becomes monopolistically competitive. The equilibrium of the limiting economy depends on the limit of the ratio of open-source and proprietary firms' investments in R&D. Even though free-riding becomes more important as the number of firms increases, collaboration becomes more important, too, so either type of firm may have an advantage.

In the basic model, we find that when the degree of public good of the investment is at its maximum level (all investment is shared), the effects of free-riding and collaboration are perfectly balanced, and the equilibrium of the limiting economy has both types of firms. In this case, as the degree of horizontal differentiation decreases, the aggregate market share of open-source firms decreases, but the proportion of open-source firms in the total of firms increases. Thus, the equilibrium has fewer but bigger proprietary firms. On the other hand, when the degree of public good of the investment is less than maximal, collaboration dominates free-riding and all firms become open source. Thus we find conditions under which *large numbers favor cooperation*; that is, open source does not disappear as the number of firms grows.

Finally, in the model with lower differentiation for open-source firms, we find that if the difference in the degree of substitution between open-source and proprietary firms is large enough (so that it compensates for the positive effects of collaboration), the limiting economy has equilibria with only proprietary firms.

The model and the results are important for a variety of reasons. First, we endogenize the decision of for-profit firms to participate in open-source projects, and the equilibrium industry structure under coexistence. Second, we show market forces and incentives may lead to an asymmetric market structure, even though all firms are ex-ante symmetric. Third, we obtain conditions under which open source can overcome free-riding and produce a good of high quality, even without coordination of individual efforts. Finally, the model allows an analysis of welfare and optimal policy.

Even though our model is specially designed to analyze open source, it has wider applicability. In particular, it can be used to analyze industries in which firms cooperating in R&D coexist with firms developing technologies on their own. In section 1.1, we discuss how this paper relates with the literature of cooperation in R&D.

The main contribution of this paper is a tractable model of competition between profit-maximizing open-source and proprietary firms. As such, the model captures the main ingredient shaping the decision to share technologies with rivals or not: the trade-off between free-riding (appropriability) and collaboration (duplication of effort). We believe our paper is an important first step in the analysis of the behavior of profit-maximizing open-source firms.

In section 1.1, we present a detailed analysis of the literature. In section 2, we introduce the basic model, which we solve in section 3. In section 4, we study the effects of free entry, and the equilibrium of the limiting economy. In section 5, we present an analysis of social welfare and optimal government policy. In section 6, we study a model with lower differentiation between open-source products. Finally, in section 7, we discuss the main implications of our analysis and present directions for further research.

1.1. Related literature

The first papers on open source were concerned with explaining why individual developers contribute to open-source projects, apparently for free (see Lerner and Tirole, 2005; von Krogh and von Hippel, 2006, for excellent surveys). The initial answers were altruism, personal gratification, peer recognition, and career concerns.

Lerner and Tirole (2001, 2002, 2005) identify directions for further research. Some of the questions related to the present paper are as follows: (i) What are the incentives of for-profit firms to participate in open source? (ii) What development model provides higher quality and welfare? (iii) How does the competitive environment influence open source? More importantly, these authors remark that direct competition between proprietary and open-source firms has received little attention. For more recent surveys, see Maurer and Scotchmer (2006) and Fershtman and Gandal (2011).

Early papers addressing competition between the two paradigms studied duopoly models of a profit-maximizing proprietary firm and a community of not-for-profit/non-strategic open-source developers, selling at zero price (Mustonen, 2003; Bitzer, 2004; Gaudeul, 2007; Casadesus-Masanell and Ghemawat, 2006; Economides and Katsamakas, 2006). In these papers, however, open-source firms have no profits, and the choice of development model is exogenous. Introducing profit-seeking open-source firms is important because doing so allows us to analyze the incentives for investing in R&D and the decision to become open source.

Later papers introduced profit-maximizing open-source firms (Johnson, 2002; Henkel, 2004; Bessen, 2006; Schmidtke, 2006; Haruvy, Sethi, and Zhou, 2008; Casadesus-Masanell and Llanes, 2011) but assumed an exogenous market structure. Other papers study industry dynamics when open-source and proprietary firms compete, but assume open-source projects are formed by altruistic contributors (Athey and Ellison, 2010) or by non-strategic firms (Arora and Bokhari, 2007). Likewise, recent papers show the organizational structure of open-source projects may lead to better development incentives than the organizational structure of proprietary firms (Johnson, 2006; Polanski, 2007; Niedermayer, 2007), but study each model in isolation and do not study direct competition between the two paradigms.

The main contributions of our paper are (i) to analyze an oligopoly model with direct competition between for-profit open-source and proprietary firms, in which (ii) the choice of development model is endogenous, and (iii) the market structure is determined endogenously as a result of firms' decisions. In this sense, the closest papers to ours are Jansen (2009) and von Engelhardt (2010).

Jansen (2009) studies a duopoly model with Cournot competition, in which firms may choose to share their knowledge to signal a low-cost position, thereby reducing competition. In contrast with Jansen, we study an oligopoly with n firms, consider Bertrand competition, and focus on the effects of technology sharing on investment incentives. Von Engelhardt (2010) studies a Cournot oligopoly in which firms may invest in open-source software, proprietary software, or both. The focus of von Engelhardt's paper differs from ours, because he is more interested in studying the effects of the type of open-source license on the equilibrium. For most of his analysis, von Engelhardt focuses on studying symmetric equilibria (i.e., all firms are of the same type), but he also presents simulation results that show that under coexistence, proprietary firms are larger than open-source firms. We provide theoretical results that formalize these findings.

Finally, our paper is also related to the literature of cooperation in R&D and research joint ventures. A first strand of papers analyzed the effects of sharing R&D on the incentives to perform such investments (D'Aspremont and Jacquemin, 1988; Kamien, Muller, and Zang, 1992; Suzumura, 1992). In particular, Kamien, Muller, and Zang show free-riding incentives are so strong that a joint venture in which firms share R&D but do not coordinate their R&D levels has a lower total investment than the individual investment of each of these firms when there is no cooperation in R&D. We show this result can be reversed when firms profit from the sale of complementary private goods.

A second strand of papers analyzed the endogenous formation of research coalitions. Bloch (1995) presents a model in which firms decide sequentially whether to join the association, and compete in quantities after forming associations. In equilibrium, two associations are formed. However, firms do not decide their optimal investments in R&D, so this model cannot be used to analyze the free-riding incentives created by association. Poyago-Theotoky (1995) and Yi and Shin (2000) assume firms set their R&D levels cooperatively after associating, and show that firms in the joint venture invest more in R&D, and have higher profits than outsiders. We show this result is reversed when firms do not coordinate their R&D levels.

A third strand of papers analyzed the endogenous determination of spillovers among firms conducting R&D. Katsoulacos and Ulph (1998) show that firms selling complementary goods may choose maximal spillovers (i.e., decide to be open source), even when they make their decisions non-cooperatively. However, firms are not competing in the same industry. In our model, firms are direct competitors in the markets for the primary and complementary goods. Amir, Evstigneev, and Wooders (2003) present a duopoly model in which firms cooperatively set their R&D levels and the strength of the spillover. In their model, firms choose maximal spillovers, because they make their decisions cooperatively.

As can be seen, the literature of cooperation in R&D is an important precedent for our paper. Nevertheless, to the best of our knowledge, previous papers have not analyzed the case of endogenous formation of a coalition cooperating in R&D, when R&D levels are determined non-cooperatively. In particular, our contribution to this literature is the result that the equilibrium in which some firms decide to cooperate and others do not is characterized by an asymmetric market structure, in which firms cooperating in R&D have smaller market shares.

2. The model

2.1. Technology

The model has n firms selling packages composed of a primary good (which is potentially open-source) and a complementary private good. This assumption fits particularly well cases in which the complementary good is essential (or almost essential) for the primary good, such as embedded systems in electronic devices (mobile phones, DVD and MP3 players, smartphones, tablet PCs, medical equipment, printers, etc.), server software, enterprise solutions, IT technical support and consulting services.

Firms may improve the quality of their packages by investing in R&D. Let x_i be the investment in R&D of firm i. The cost of investment is $c\,x_i$, which is an endogenously determined fixed cost. The marginal cost of producing packages is zero.

Firms may choose to develop their primary goods under the open-source or proprietary development models, which we denote as O and P, respectively. O firms collaborate in the development of the primary good, and quality is given by $q_i = \alpha\ln(\Sigma_{i\in O}\,x_i) + (1-\alpha)\ln(x_i)$, where $\Sigma_{i\in O}\,x_i$ is the sum of investments of O firms, and $\alpha \in [0, 1]$ represents the degree of public good of the investment in R&D. P firms invest individually, and quality is given by $q_i = \ln(x_i)$.

Our specification for the quality of O packages implies O firms benefit more from their own investment than from the investment of other firms in the open-source project. Several reasons motivate this assumption. First, O firms have incentives to contribute code to sections of the program that benefit them more than other firms. Second, even though firms are compelled to share their improvements to the primary good, the availability of these improvements to other firms may be delayed. Third, a learning effect may be present through which firms gain valuable knowledge and expertise when they increase their participation in the open-source project, and thus are able to offer a better complementary good.

Finally, α may also indicate the degree of restrictiveness of the open-source license. Restrictive licenses, such as the General Public License, force developers to share their contributions to the code if they distribute the modified program. Permissive licenses, such as the BSD License, on the other hand, allow developers to keep their contributions private. Therefore, as α increases, the open-source license becomes more restrictive and a higher fraction of the source code is shared.[2]

2.2. Preferences

A continuum of consumers exists. Each consumer has income y and buys one package. Consumer j's indirect utility from consuming package i is

$$v_{ij} = q_i + y - p_i + \varepsilon_{ij}, \tag{1}$$

where q_i is the quality of the package of firm i, p_i is its price, and ε_{ij} is an idiosyncratic shock (unobservable by firms) representing the heterogeneity in tastes between consumers. This specification for preferences allows for vertical (q_i) and horizontal (ε_{ij}) product differentiation.

Each consumer *observes* prices and qualities and then chooses the package that yields the highest indirect utility. The total mass of consumers is 1, so aggregate demands are equivalent to market shares. To obtain closed-form solutions for the demands, we make the following assumption, which corresponds to the multinomial logit model (McFadden, 1974)[3]:

Assumption 1. *The idiosyncratic taste shocks ε_{ij} are i.i.d. according to the double exponential distribution:*

$$\Pr(\varepsilon_{ij} \leq z) = \exp\left(-\exp\left(-\nu - \delta\, z\right)\right)$$

where ν is Euler's constant ($\nu \approx 0.5772$) and δ is a positive constant.

[2]We are grateful to a referee for suggesting this interpretation.

[3]The logit is a common model in discrete choice theory (Ben-Akiva and Lerman, 1985), and has been widely used in econometric applications (see Train, McFadden, and Ben-Akiva, 1987, and references therein), in marketing (McFadden, 1986), and in theoretical work (see Besanko, Perry, and Spady, 1990; Anderson and de Palma, 1992; Anderson and Leruth, 1993). See Anderson, de Palma, and Thisse (1992) for a detailed presentation of its main properties.

Under Assumption 1, the market share (demand) of firm i is

$$s_i = \frac{\exp\left(\delta\left(q_i - p_i\right)\right)}{\sum \exp\left(\delta\left(q_i - p_i\right)\right)}. \tag{2}$$

The taste shocks have zero mean and variance $\pi^2/(6\,\delta^2)$. As δ increases, consumers become less differentiated and the degree of horizontal differentiation among varieties decreases. To guarantee the existence of a symmetric equilibrium, we need to assume enough horizontal differentiation exists relative to vertical differentiation (Yarrow, 1989; Anderson, de Palma, and Thisse, 1992). In the proof of Proposition 2, we show a sufficient condition is $\delta \leq 1$, which we assume throughout the paper.

2.3. Game and equilibrium concept

The model is a two-stage non-cooperative game. The players are the n firms. In the first stage, firms decide their type (O or P), and in the second stage, they choose investment and prices (x_i, p_i).

Given investments (quality) and prices, each consumer chooses her optimal package. These decisions are summarized by consumer demands (s_i) and embedded into the firms' payoffs: $\pi_i = s_i\, p_i - c\, x_i$.

The equilibrium concept is subgame perfect equilibrium, and we focus on symmetric equilibria; that is, all firms deciding to be of the same type in the first stage play the same equilibrium strategy in the second stage.

3. Solution of the model

3.1. Second stage

Let n_o be the number of firms deciding to be O in the first stage. In the second stage, firms choose p_i and x_i to maximize $\pi_i = s_i\, p_i - c\, x_i$, taking as given the decisions of other firms. Working with the first-order conditions and imposing symmetry, we obtain the optimal price:

$$p_i = \frac{1}{\delta\,(1 - s_i)}, \tag{3}$$

and the optimal investment in R&D for O and P firms:

$$x_o = \frac{1}{c}\, s_o \left(1 - \alpha\, \frac{n_o - 1}{n_o(1 - s_o)}\right), \tag{4}$$

$$x_p = \frac{1}{c}\, s_p. \tag{5}$$

The term inside the parenthesis of (4) represents free-riding. If $s_o = s_p$, O firms would invest less than P firms because they can appropriate a smaller fraction of their investment.

From (2), we can get the ratio of market shares s_o/s_p. Introducing equations (3) to (5), taking logs and rearranging terms, we obtain

$$(1 - \delta)\,\ln\left(\frac{s_o}{s_p}\right) + \frac{1}{1 - s_o} - \frac{1}{1 - s_p} = \delta\,\ln\left(1 - \alpha\,\frac{n_o - 1}{n_o(1 - s_o)}\right) + \alpha\,\delta\,\ln\left(n_o\right). \tag{6}$$

Equation (6) shows the difference in market shares depends on the resolution of the conflict between *free-riding* and *collaboration*. The left-hand side is increasing in s_o and decreasing in s_p, so the difference in market shares will increase if the right-hand side does. The first term on the right-hand side arises from the difference in individual investments (free-riding). The second term arises because individual investments in open source are multiplied by the number of O firms (collaboration).

The trade-off between free-riding and collaboration is determined by α and n_o. On one hand, as α increases, the degree of public good of the investment in R&D increases, and thus the individual investments of O firms (x_o) decrease. On the other hand, as α increases, the joint investment of O firms ($n_o\, x_o$) has a greater effect on quality. Likewise, as n_o increases, the public-good problem becomes more important (more firms are sharing), but collaboration also becomes more important (more firms are collaborating). Moreover, α and n_o are complementary. The effects of a higher n_o on free-riding and collaboration become more important when α is higher.

The second-stage equilibrium is completely characterized by (6) and the condition that the sum of the market shares is equal to 1:

$$n_o\, s_o + (n - n_o)s_p = 1. \tag{7}$$

Proposition 1. *A second-stage equilibrium exists and is unique. Given n_o, the equilibrium market shares solve (6) and (7).*

All proofs are relegated to the Appendix. In what follows, we study the comparative statics of the second-stage equilibrium. In Lemma 1, we present a simple condition to determine which type of firm will have higher market share (quality and price).

Lemma 1. *$s_p > s_o$ if $\alpha > \hat{\alpha}(n_o, n)$, and $s_p < s_o$ if $\alpha < \hat{\alpha}(n_o, n)$, where $\hat{\alpha}(n_o, n)$ is increasing in n_o and n and solves:*

$$\alpha\, \frac{n_o^\alpha}{n_o^\alpha - 1}\, \frac{n_o - 1}{n_o} = \frac{n-1}{n}.$$

The comparison of prices and quality is equivalent to the comparison of market shares: if $s_o > s_p$, then $p_o > p_p$ and $q_o > q_p$, and vice versa. Lemma 1 provides an important result: as n_o or n increase, the region of parameters for which O firms have higher market share (quality and price) than P firms increases.

Lemmas 2 and 3 analyze the effects of changes in α and δ on s_o. The effects on s_p have the opposite sign.

Lemma 2. *A threshold $\alpha_s \in (0, \hat{\alpha})$ exists such that s_o is increasing in α if $\alpha < \alpha_s$, and decreasing in α if $\alpha > \alpha_s$.*

Lemma 2 implies the graph of s_o with respect to α (the degree of public good of the investment) is hump-shaped. When α is close to zero, the investment of O firms is mostly private, and individual investments are close to the investments of P firms. Therefore, the positive effects of collaboration are more important than the negative effects of free-riding. For high values of α, free-riding becomes more important and the difference in individual investments between O and P firms increases. Therefore, for large α, free-riding dominates collaboration.

Lemma 3. *s_o is increasing in δ if $\alpha < \hat{\alpha}(n_o, n)$, and decreasing in δ if $\alpha > \hat{\alpha}(n_o, n)$.*

Lemma 3 shows the effect of a higher δ depends on the value of α. When δ increases, vertical differentiation becomes more important relative to horizontal differentiation. As a consequence, investing in R&D has a greater effect on demand, which benefits firms with higher-quality products. If $\alpha < \hat{\alpha}$, O firms have higher-quality products; therefore, their market share increases relative to the market share of P firms. The opposite happens when $\alpha > \hat{\alpha}$.

3.2. First stage

In the first stage of the game, firms choose to be O or P, taking as given the decisions of the rest of the firms and forecasting their equilibrium payoffs in the second stage. Let $\pi(n_o)$ be the second-stage equilibrium payoffs when n_o firms decide to be O. Replacing the second-stage equilibrium values of prices and investments for both types of firms, we obtain

$$\pi_o(n_o) = \frac{s_o}{1-s_o}\left(\frac{1}{\delta} - (1-s_o) + \alpha\,\frac{n_o-1}{n_o}\right), \tag{8}$$

$$\pi_p(n_o) = \frac{s_p}{1-s_p}\left(\frac{1}{\delta} - (1-s_p)\right), \tag{9}$$

where $s_o = s_o(n_o)$ and $s_p = s_p(n_o)$ are the second-stage equilibrium market shares. Equilibrium profits are always positive, given that α, δ, s_o, and s_p are all between 0 and 1. Comparing equations (8) and (9), we can see collaboration has a direct effect on profits (third term inside the parenthesis of equation 8): if $s_o = s_p$, the investment of O firms is lower than the investment of P firms ($x_o = x_p/n_o$), and this effect becomes larger as α increases.

A number n_o of firms in the open-source project is an equilibrium if and only if $\pi_o(n_o) \geq \pi_p(n_o-1)$ and $\pi_p(n_o) \geq \pi_o(n_o+1)$. D'Aspremont, Jacquemin, Gabszewicz, and Weymark (1983) called these conditions internally stable and externally stable coalition conditions. The first inequality says firms deciding to be O cannot gain by deviating and becoming P. The second inequality is a similar condition on the decision of being P. Using the function $f(n_o) = \pi_o(n_o) - \pi_p(n_o-1)$, equilibrium conditions can be restated as $f(n_o) \geq 0$ and $f(n_o+1) \leq 0$.

The equilibrium may be such that both types of firms coexist (interior equilibrium) or all firms choose to be of the same type. $n_o = 0$ is always an equilibrium, so we focus on equilibria with $n_o \geq 1$. For $n_o = 1$ to be an equilibrium, we need $f(2) \leq 0$. Likewise, for $n_o = n$ to be an equilibrium, we need $f(n) \geq 0$.

Figure 1 shows an example of the $f(n_o)$ schedule when $\alpha = 1$, $\delta = 0.7$, and $n = 10$. In this case, six firms are open source in equilibrium.

When firms choose between O or P, they compare the relative benefits of collaboration and secrecy. Two elements are associated with this trade-off. On the one hand, free-riding and collaboration affect the equilibrium market shares, as analyzed in section 3.1. On the other hand, O firms have a lower investment cost. Being P will be more profitable than being O only if free-riding is strong enough to overcome the positive effects of collaboration. Proposition 2 characterizes the equilibrium of the game.

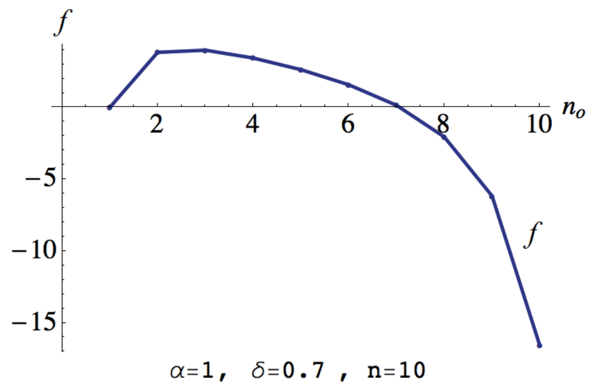


Figure 1: Equilibrium number of firms in open source.

Proposition 2. *A subgame perfect equilibrium exists. Given $n > 3$ and δ, thresholds $0 < \underline{\alpha} < \bar{\alpha} < 1$ exist such that:*

(i) If $\alpha > \bar{\alpha}$, both types of firms coexist and P firms have higher quality and market share than O firms.

(ii) If $\underline{\alpha} < \alpha \leq \bar{\alpha}$, all firms decide to be O, but a P firm would have higher quality and market share.

(iii) If $\alpha \leq \underline{\alpha}$, all firms decide to be O, and a P firm would have lower quality and market share.

Proposition 2 shows three equilibrium regions exist. When α is large, the degree of public good of the investment in R&D is high, and free-riding is important, which leads to low individual investments for O firms (see Lemma 2). As a consequence, P firms have an advantage over O firms in terms of market share and price. On the other hand, O firms benefit from lower development costs. Therefore, there is room for both types of firms in equilibrium: some firms choose to be P and have a high investment in R&D to benefit from high market shares and prices, and other firms choose to be O to benefit from low development costs. For intermediate values of α, the market-share advantage of P firms is not enough to compensate the higher development costs. Therefore, all firms decide to be O, but a P firm would produce a good of higher quality. In this case, open source is preventing the entry of a product of better quality. Finally, when α is small, the positive effects of collaboration on investment incentives are stronger than the negative effects of free-riding (see Lemma 2), and the total investment in the open-source project is larger than the investment of a P firm. Basically, when alpha is very close to 0, but positive, the individual investments of O and P firms are very similar because very little is shared. In this case, firms benefit from having some level of cooperation, and that is why all firms choose to be O.

Figure 2 shows the equilibrium regions for different values of α and n, when $\delta = 1$. The area corresponding to equilibria with coexistence first increases but then decreases with n, which means large numbers favor cooperation, even without coordination of individual investments. In the following section, we elaborate on this result.

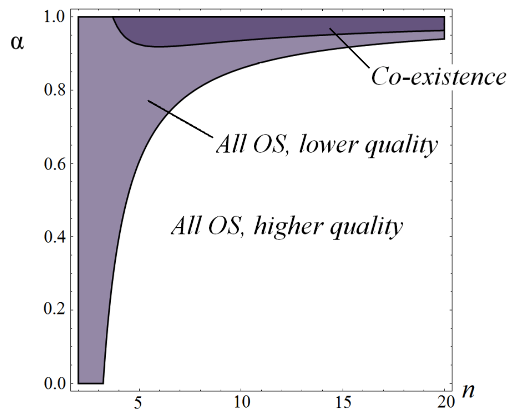


Figure 2: Equilibrium regions.

In the basic model, no equilibrium exists with only P firms. O and P firms are symmetric in all aspects, except for the fact that O firms share their investments in R&D. In section 6, we introduce an additional difference: given that O firms base their packages on the same primary good, O products tend to be more similar than P products. We show that if the price elasticity of O products is not large enough, strong competition leads to low prices for open-source packages, in which case the open-source model becomes less desirable and an equilibrium with only P firms is possible.

Likewise, results may change if O firms can direct their investment towards the shared code or towards a directly appropriable complementary good, such as a proprietary application. If the two types of investment are complementary and O firms have low incentives to invest in the primary good, investment in the complementary good will be low as well, in which case an equilibrium with only P firms may exist. We believe this extension is an interesting direction for further research.

4. Entry

In previous sections, we assumed a fixed number of firms. Implicitly, we were assuming an *exogenous* cost of entry, in addition to the *endogenous* cost of developing the primary good. Concretely, suppose firms decide to enter the industry before deciding to become O or P. As the cost of entry decreases, the number of firms in equilibrium increases and we move along the curve of Figure 2.

The fixed cost of entry may be a consequence of entry barriers in the market of complementary goods. The market of smartphones, for example, has high barriers to entry. Even though the Android operating system is free for firms wanting to develop a smartphone, this industry has few producers, because developing the hardware that goes with the software is expensive. In other markets of complementary goods, however, the cost of entry is small and a large number of firms is present. For example, the cost of entering the market of software training, support, and customization is relatively small, and a large number of independent software programmers provide such services. In fact, the open-source movement started precisely among these independent programmers.

For this reason, studying the case of a small cost of entry is of interest.[4] In our model, as the cost of entry goes to zero, the number of firms in the industry goes to infinity and the industry becomes monopolistically competitive. As Besanko, Perry, and Spady (1990) show in their analysis of the logit model of monopolistic competition, price converges to a constant margin over marginal cost, which depends on the substitution parameter δ. The key assumption is horizontal differentiation, which allows any firm to enter and supply a differentiated product (Dixit and Stiglitz, 1977).[5] Proposition 3 characterizes the equilibrium of the limiting economy.

Proposition 3. *A subgame perfect equilibrium of the limiting economy ($n \to \infty$) exists and is unique. The equilibrium is characterized as follows:*

(i) If $\alpha < 1$, all firms decide to be O.
(ii) If $\alpha = 1$, both types of firms coexist, the ratio of market shares (s_o/s_p) is $1-\delta$, the aggregate market share of O firms ($n_o\, s_o$) is $1-(1-\delta)^{\frac{1-\delta}{\delta}}$ and the proportion of O firms (n_o/n) is $\frac{1-(1-\delta)^{(1-\delta)/\delta}}{\delta-(1-\delta)^{(1-\delta)/\delta}}$.

Proposition 3 shows the incentives to participate and to invest in an open-source project do not disappear when the number of firms goes to infinity. This result is somewhat surprising because as n_o increases, free-riding intensifies and individual incentives to invest in open source decrease. In the limit, we would expect the incentives to invest in open source to disappear completely. However, as n_o increases, collaboration between O firms also intensifies (individual investments of O firms are multiplied by a larger factor), which compensates for the negative effects of free-riding.

As $n \to \infty$, investment in R&D and market share converge to zero for *both* O and P firms, as firms become infinitesimally small. However, studying whether investments and market shares converge faster to zero for O or P firms (i.e., determining the limits of the ratios of investments and market shares) is interesting because doing so allows us to determine if large numbers favor cooperation in R&D.

As $n_o \to \infty$, the ratio of individual investments (x_o/x_p) converges to $1-\alpha$. When α is close to 1, individual incentives to invest in the open-source good are low. However, as long as $\alpha < 1$, the ratio x_o/x_p is strictly positive and the individual investment of O firms is multiplied by a factor that goes to infinity. Therefore, $\frac{n_o\, x_o}{x_p} \to \infty$ and $s_o/s_p \to \infty$, which means all firms choose to be O.

When $\alpha = 1$, on the other hand, $n_o \to \infty$, $x_o/x_p \to 0$, and their product converges to a constant. In equilibrium, the ratio of market shares converges to $1-\delta$ and the ratio of profits converges to 1, so both types of firms coexist. As in Proposition 2, in the equilibrium with coexistence, P firms are larger than O firms.

In the equilibrium with coexistence, the aggregate market share of O firms is decreasing in δ, but the proportion of O firms is *increasing* in δ. As δ increases, two effects

[4] We are grateful to a referee for suggesting this extension.

[5] In the logit model, the support for consumer tastes (ε_{ij}) is the real line, which means a new firm can always grab *some* market share with a small investment in R&D, even under Bertrand competition.

occur: competition intensifies, which lowers mark-ups; and vertical differentiation becomes more important, which increases the returns to investment. As a consequence, P firms become larger, the aggregate market share of O firms becomes smaller, and there is room for fewer P firms.

5. Welfare analysis

One of the advantages of the logit model is that it can be used to construct a representative consumer whose utility embodies the aggregate behavior of the continuum of users (Anderson, de Palma, and Thisse, 1992).

Let s_i be the quantities of each variety the representative consumer consumes, and let $\sum s_i = 1$. Total income is y, and z represents consumption of the numeraire. The utility of the representative consumer is

$$U = \sum q_i\, s_i - \frac{1}{\delta} \sum s_i \ln(s_i) + z.$$

The utility function embodies two different effects. The first term represents the direct effect from consumption of the n varieties, in the absence of interactions. The second term introduces an entropy effect, which expresses the representative consumer's preference for variety.

The utility function is quasilinear, which implies transferable utility. Thus, social welfare is the sum of consumer utility and firm profits:

$$W = \sum q_i\, s_i - \frac{1}{\delta} \sum s_i \ln(s_i) + y - \sum c\, x_i. \tag{10}$$

The Social Planner's problem is to maximize (10) subject to $\sum s_i = 1$. The Social Planner will always have all firms sharing their investment in R&D. Also, given the concavity and symmetry of the utility function, the social planner will set $s_i = 1/n$ for all i. To determine the optimal investment, the Social Planner maximizes

$$W = \alpha\, \ln(n\, x^*) + (1-\alpha)\, \ln(x^*) + \frac{1}{\delta} \ln(n) + y - n\, c\, x^*,$$

which leads to an optimal investment equal to $x^* = 1/(c\, n)$.

In the market equilibrium, product quality is suboptimal regardless of the number of O and P firms: O firms are subject to free-riding, which leads to a suboptimal investment in R&D, but P firms do not share their improvements on the primary good, generating an inefficient duplication of effort.

5.1. Government policy

Now we turn to an analysis of government policy. We will show the first best can be achieved by using a tax-subsidy scheme. The cost of R&D for O is $c_o = (1-\kappa)\, c$, where κ is a proportional subsidy on the investment of O firms. This subsidy, in turn, is financed by proportional or lump-sum taxes paid by consumers.

The ratio of investments of O and P firms becomes

$$\frac{x_o}{x_p} = (1-\kappa)^{-1}\,\frac{s_o}{s_p}\,\left(1-\alpha\,\frac{n_o-1}{n_o}\,\frac{1}{1-s_o}\right),$$

and the equation characterizing the equilibrium market shares becomes

$$(1-\delta)\,\ln\left(\frac{s_o}{s_p}\right)+\frac{1}{1-s_o}-\frac{1}{1-s_p}=\delta\,\ln\left(1-\alpha\,\frac{n_o-1}{n_o(1-s_o)}\right)+\alpha\,\delta\,\ln\left(n_o\right)-\delta\ln\left(1-\kappa\right).$$

An increase in the subsidy increases the difference in investments and market shares between O and P, and also decreases the cost of investment for O, so firms are more tempted to become O. Lemma 4 shows that if the subsidy is high enough, O firms will have a higher market share than P in a second-stage equilibrium.

Lemma 4. *$s_o > s_p$ in a second-stage equilibrium if and only if*

$$\kappa > 1-\left(1-\alpha\frac{n}{n-1}\,\frac{n_o-1}{n_o}\right)n_o^{\alpha}.$$

In particular, if $\kappa > 1-\left(1-\alpha n(n-2)(n-1)^{-2}\right)(n-1)^{\alpha}$, then $s_o > s_p$ for $n_o = n-1$; therefore, all firms want to be O. Proposition 4 shows the optimal policy.

Proposition 4. *The optimal subsidy is $\kappa^* = \alpha$, which attains the first-best levels of investment. In equilibrium, all firms decide to be O.*

The subsidy has a double effect: it increases the investment of O firms, and it encourages P firms to become O (to share R&D). The optimal subsidy is increasing in the degree of public good of the investment in R&D. In other words, the subsidy should be higher for projects for which the direct appropriability of the investment in R&D of O firms is not very high.

Finally, note that in our model, lump-sum or proportional taxes are equivalent, because each consumer buys one product; therefore, proportional taxes do not affect the quantities sold. Thus financing the subsidy with proportional taxes does not cause a deadweight loss and the policy maker can achieve the first best.

6. Lower differentiation for open-source products

Given that O packages share the same primary good, they are likely to be more similar to each other than P packages. To introduce this difference in the degree of substitution, we use a nested logit model (Ben-Akiva, 1973), which adds an element of *endogenous* horizontal differentiation to the trade-off between collaboration and secrecy. By becoming P, firms are able to differentiate their product more than O firms.

The main consequences are that (i) the equilibrium number of firms in O is smaller than in the previous model, (ii) equilibria with only P firms exist, and (iii) parameter values exist that lead to multiple equilibria.

Consumers are heterogeneous in two different dimensions: they have idiosyncratic tastes for the primary good and idiosyncratic tastes for the complementary good. The relative strength of these two forces drives the differences in substitution. Following the nested logit representation of Cardell (1997), consumer j's indirect utility from consuming package i, based on primary good k, is

$$v_{ikj} = q_i + y - p_i + \eta_{kj} + (1-\sigma)\,\varepsilon_{ij},$$

where q_i is defined as in section 2, η_{kj} is a primary good idiosyncratic component, and $\sigma \in [0,1]$ weighs the different idiosyncratic components. Assumption 2 replaces Assumption 1 for the standard logit case.

Assumption 2. *The idiosyncratic components ε_{ij}, corresponding to complementary good i, are i.i.d. according to the double exponential distribution with scale parameter δ. The idiosyncratic components η_{kj}, corresponding to primary good k, are i.i.d. according to a distribution such that $\eta_{kj} + (1-\sigma)\,\varepsilon_{ij}$ is distributed double exponential with scale parameter δ.*

Assumption 2 implies the horizontal differentiation term $\eta_{kj} + (1-\sigma)\,\varepsilon_{ij}$ has the same distribution as ε_{ij} in the previous model. The variance of η_{kj} is $\sigma(2-\sigma)\pi^2/(6\,\delta^2)$. Cardell shows a unique distribution for η_{kj} exists such that Assumption 2 holds.

Parameter σ determines the relative strength of the horizontal differentiation forces. As σ increases, consumers become more differentiated in their tastes for the primary good and less differentiated in their tastes for the complementary good. When $\sigma = 0$, consumers only have idiosyncratic preferences for the complementary good, and the model becomes the standard logit model of previous sections. When $\sigma = 1$, consumers only have idiosyncratic preferences for the primary good, and O firms sell a homogeneous good.

The proportion of consumers choosing open-source variant i can be decomposed in the following way:

$$s_i = s_{i|o}\, S_o, \tag{11}$$

where S_o is the aggregate market share of the open-source primary good, and $s_{i|o}$ is the share of variant i within the open-source project.

Under Assumption 2, i's market share within the open-source project depends on its individual contribution to the project:

$$s_{i|o} = \frac{\exp\left(\delta \dfrac{(1-\alpha)\,\ln x_i - p_i}{(1-\sigma)}\right)}{\sum_{i\in O} \exp\left(\delta \dfrac{(1-\alpha)\,\ln x_i - p_i}{(1-\sigma)}\right)}.$$

The aggregate market share S_o depends on the average value of the O varieties (the expected value of the maximum of the utilities), V_o:

$$S_o = \frac{\exp(\delta\, V_o)}{\exp(\delta\, V_o) + \sum_{i\in P} \exp(\delta(q_i - p_i))},$$
$$V_o = \frac{(1-\sigma)}{\delta}\ \ln\left(\sum_{i\in O} \exp\left(\frac{\delta(q_i - p_i)}{(1-\sigma)}\right)\right).$$

P nests are composed only of one P product, so the average value of the nest is the value of its only component. Therefore, the market share of a P firm is simply

$$s_i = \frac{\exp(\delta(q_i - p_i))}{\exp(\delta\, V_o) + \sum_{i\in P} \exp(\delta(q_i - p_i))}. \tag{12}$$

Further intuition on the substitution patterns implied by the assumptions on preferences can be obtained by computing the demand sensitivity to changes in price. The slope of the demand function for a P firm is

$$\frac{\partial s_i}{\partial p_i} = -\delta s_i(1 - s_i),$$

and for an O firm, it is

$$\frac{\partial s_i}{\partial p_i} = -\delta s_i\left(\frac{1}{1-\sigma} - \frac{\sigma}{1-\sigma} s_{i|o} - s_i\right).$$

These expressions summarize the substitution patterns in the nested logit model: O firms face a more elastic demand than P firms, and the price elasticity for O firms is increasing in σ. If $\sigma = 0$, the slope of the demand is the same for O and P firms, as in the standard logit model. If $\sigma = 1$, the slope of the demand for O firms goes to infinity.

The optimal price and investment of P firms have the same functional forms as before. The optimal price and investment of O firms become

$$p_o = \frac{1}{\delta\left(1 - s_o + \frac{\sigma}{1-\sigma}\,\frac{n_o - 1}{n_o}\right)}, \tag{13}$$

$$x_o = \frac{1}{c}\, s_o\left(1 - \frac{\alpha}{1-\sigma}\,\frac{(n_o - 1)/n_o}{\left(1 - s_o + \frac{\sigma}{1-\sigma}\,\frac{n_o - 1}{n_o}\right)}\right). \tag{14}$$

Conditional on market shares, optimal prices for O firms are decreasing in σ, whereas optimal prices for P firms are independent of σ.

From (12) and (11), we obtain the ratio of market shares s_o/s_p. Introducing prices and investments, taking logs and rearranging terms, we obtain

$$(1-\delta)\ln\left(\frac{s_o}{s_p}\right) + \frac{1}{1 - s_o + \frac{\sigma}{1-\sigma}\,\frac{n_o-1}{n_o}} - \frac{1}{1-s_p} \tag{15}$$
$$= \delta\,\ln\left(1 - \frac{\alpha}{1-\sigma}\,\frac{(n_o-1)/n_o}{\left(1 - s_o + \frac{\sigma}{1-\sigma}\,\frac{n_o-1}{n_o}\right)}\right) + (\alpha\,\delta - \sigma)\,\ln(n_o).$$

As in the standard logit case, to guarantee the existence of a symmetric equilibrium we need enough horizontal differentiation relative to vertical differentiation. We assume $\sigma \leq 1-\delta$, which is a sufficient condition. Proposition 5 summarizes the equilibrium of the second-stage of the game.

Proposition 5. *A second-stage equilibrium for the nested model exists and is unique. Given n_o, the equilibrium market shares solve (15) and (7).*

Comparing equations (6) and (15), we can see the higher substitution between O varieties has three effects on equilibrium market shares. First, a lower investment of O firms occurs due to the lower return to investment (first term on the right-hand side of 15). Second, the higher substitution directly and negatively affects the average value of the complementary good (second term on the right-hand side of 15). Consumers care for variety; therefore, the value of choosing an O package decreases when the complementary good becomes less differentiated. Third, the equilibrium price of O firms is smaller because of the higher substitution (second term on the left-hand side of 15). The first two effects tend to reduce the market share of O relative to P, and the third effect tends to increase it.

To solve the first stage of the game, we calculate $f(n_o) = \pi_o(n_o) - \pi_p(n_o - 1)$, where $\pi(n_o) = p_i\, s_i - c\, x_i$. Equilibrium conditions are the same as in the standard logit case.

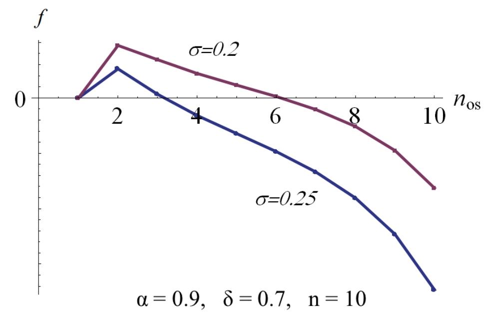
Figure 3 shows the graph of $f(n_o)$ for different parameter values. We can make three interesting observations. First, as σ increases for given α and δ (O varieties become more similar), the equilibrium number of firms choosing O decreases (Figure 3a). Second, if σ is high enough, equilibria with only P firms exist (Figure 3b). Third, for some parameter values, the model exhibits multiple equilibria (in Figure 3c an equilibrium exists with $n_o = 2$ and another with $n_o = 10$). In this case, coordination failures may imply a desirable open-source project fails to form.

Figure 4 shows the equilibrium regions for different values of α and σ, given $\delta = 0.6$ and $n = 10$ (in case of multiple equilibria, we take the equilibrium with highest n_o). Open source will subsist if the differentiation between O varieties is high enough (σ is low enough). Also, values of α exist such that as σ increases, the equilibrium goes from all O, to coexistence and then to all P. For coexistence, we need a combination of low σ and high α. Finally, our simulations show that whenever O and P coexist, the quality and market share of P firms is larger than that of O firms, which means that the main result of the paper still holds.

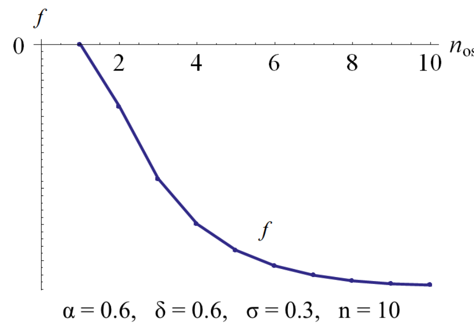
As in the case of the standard logit, studying what happens when the number of firms goes to infinity is interesting. Proposition 6 characterizes the equilibrium of the limiting economy for the nested logit model and formalizes the previous intuition on the effects of σ on the equilibrium.

Proposition 6. *A subgame perfect equilibrium of the limiting economy exists, but the model may exhibit multiple equilibria. A threshold $0 \leq \tilde{\sigma}(\alpha, \delta) < \alpha\, \delta$ exists such that*

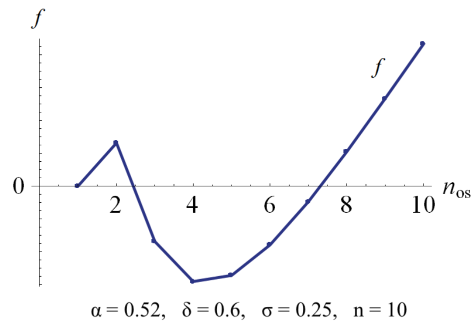
(i) If $\alpha < 1$ and $\sigma < \alpha\, \delta$, an equilibrium exists in which all firms decide to be O.



(a) Decrease in n_o when σ increases.



(b) All P Equilibrium.



(c) Multiple Equilibria.

Figure 3: Equilibrium of the nested logit model.

(ii) If $\sigma \geq \tilde{\sigma}(\alpha, \delta)$, an equilibrium exists in which all firms decide to be P.

(iii) If $\alpha = 1$ and $\sigma < \tilde{\sigma}(1, \delta)$, an equilibrium exists in which both types of firms coexist.

As σ increases, O products become more similar. When $\alpha < 1$, the ratio of equilibrium market shares depends on the factor ${n_o}^{\alpha\,\delta-\sigma}$. When $\alpha\,\delta - \sigma > 0$, the effects of collaboration in R&D are stronger than the effects of lower substitution, and the difference in market shares between O and P firms grows large as n_o goes to infinity. As a consequence, an equilibrium exists in which all firms decide to be O. When $\alpha\,\delta - \sigma \leq 0$, collaboration in R&D is not strong enough to compensate for the effects of having a lower substitution; therefore, P firms obtain an advantage in market share. Thus, no equilibrium with O firms exists. For intermediate values of σ, two equilibria exist, one in which all firms choose to be O and another in which all firms choose to be P.

Finally, when $\alpha = 1$, two types of equilibria exist. If $\sigma < \tilde{\sigma}(1, \delta)$, the unique equilibrium has both types of firms. As σ gets closer to 0, the number of O firms in equilibrium increases.[6] If $\sigma \geq \tilde{\sigma}(1, \delta)$, on the other hand, the unique equilibrium has only P firms.

[6] Unlike the case of $\sigma = 0$, when $\sigma > 0$ and $\alpha = 1$, the number of O firms in equilibrium is finite. See the proof of Proposition 6 for more details.

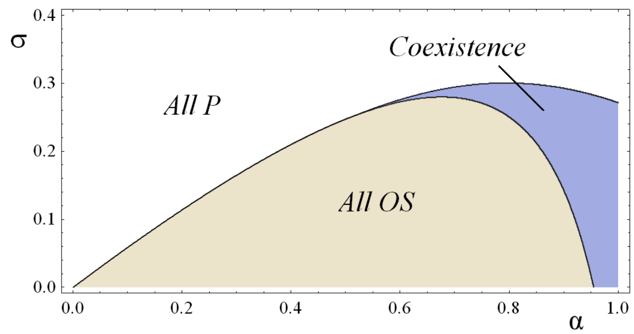


Figure 4: Equilibrium regions for the nested logit.

7. Conclusion

This paper investigates the motivations of commercial firms to participate in open source, and the implications of direct competition between open-source and proprietary firms on R&D investments and equilibrium market shares. We present a model in which firms decide whether to become open source or proprietary, and their investment in R&D and price. Both types of firms sell packages composed by a primary good (e.g., software) and a complementary private good (e.g., support and training services or hardware). The difference between both types of firms is that open-source firms share their investments in R&D, whereas proprietary firms develop their products on their own.

Our main contribution is to determine conditions under which open-source and proprietary firms coexist in equilibrium. This equilibrium is characterized by an asymmetric market structure: proprietary firms invest more in R&D and obtain a larger market share than open-source firms. Open-source firms, on the other hand, benefit from lower development costs. This result is robust to the introduction of a lower differentiation among open-source varieties. We also study a limiting economy, and show conditions under which large numbers favor cooperation in R&D.

Our model points to several important characteristics of open source. In particular, the success of open source will depend on (i) the strength of the complementarity between primary and complementary goods, (ii) the possibility to differentiate the firm's open-source variant from other open-source and proprietary products, and (iii) the degree of appropriability of investments in R&D.

The welfare analysis shows the equilibrium with coexistence is suboptimal for two reasons: too little collaboration (caused by proprietary firms) and too little investment in R&D (caused by open-source firms). We show a subsidy to open-source development can improve welfare not only because it increases the investment in R&D, but also because it encourages commercial firms to participate in open source, thereby enhancing collaboration. This finding explains the active involvement of governments in promoting open source.

Our objective was to present a tractable model analyzing the coexistence of open-source and proprietary firms. We believe our paper is an important first step in the analysis of the behavior of profit-maximizing firms in open source. Several directions for further research are possible. First, the model could be modified to study bundling and compatibility decisions. Second, consumer preferences could be modified to introduce network effects. Third, an important technological difference between open-source and proprietary firms is that the former benefit more from user innovation than the latter. In open source, users can access the source code, which allows them to customize the software program to their needs and to correct bugs at a faster rate.

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Appendix A: Proofs of Theorems in Text

Proof of Proposition 1. The first-order conditions with respect to p_i and x_i are

$$\frac{\partial \pi_i}{\partial p_i} = \frac{\partial s_i}{\partial p_i} p_i + s_i \leq 0 \qquad \text{with equality if } p_i > 0, \tag{16}$$

$$\frac{\partial \pi_i}{\partial x_i} = \frac{\partial s_i}{\partial x_i} p_i - c \leq 0 \qquad \text{with equality if } x_i > 0. \tag{17}$$

For the moment, assume $p_i > 0$ and $x_i > 0$ in equilibrium, so the first-order conditions hold with equality. Later, we will show no corner equilibria exist. Working with equation (16), we obtain the optimal price:

$$p_i = \frac{1}{\delta(1 - s_i)}. \tag{18}$$

Equation (18) holds for both types of firms (O and P). To find the optimal investment in R&D, we need to calculate $\partial s_i / \partial x_i$, which in the case of O firms, is

$$\frac{\partial s_i}{\partial x_i} = \delta\, s_i (1 - s_i) \left(\frac{\alpha}{\sum_{i \in O} x_i} \frac{1 - \sum_{i \in O} s_i}{1 - s_i} + \frac{(1 - \alpha)}{x_i} \right),$$

and in the case of P firms, is

$$\frac{\partial s_i}{\partial x_i} = \delta\, s_i(1-s_i)\frac{1}{x_i}.$$

Imposing symmetry and introducing these expressions into (17), we obtain

$$x_o = \frac{1}{c}\, s_o \left(1 - \alpha\, \frac{n_o - 1}{n_o}\, \frac{1}{1-s_o}\right), \tag{19}$$

$$x_p = \frac{1}{c}\, s_p, \tag{20}$$

and the ratio of optimal investments in equilibrium,

$$\frac{x_o}{x_p} = \frac{s_o}{s_p}\left(1 - \alpha\, \frac{n_o - 1}{n_o}\, \frac{1}{1-s_o}\right). \tag{21}$$

From (2), we obtain the ratio of market shares between O and P firms:

$$\frac{s_o}{s_p} = \exp\left(\delta(q_o - q_p + p_p - p_o)\right), \tag{22}$$

$$\ln\left(\frac{s_o}{s_p}\right) = \delta\,(q_o - q_p) + \frac{1}{1-s_p} - \frac{1}{1-s_o}, \tag{23}$$

and from the definition of q_o and q_p, we obtain

$$q_o - q_p = \ln\left(\frac{x_o}{x_p}\right) + \alpha\, \ln(n_o). \tag{24}$$

From equations (21), (23), and (24), we obtain equation (6), which is an implicit equation determining the relation of market shares between O and P firms in equilibrium. This equation, together with the equation establishing that the sum of the market shares is equal to 1, completely characterizes the equilibrium.

Now we will show no corner solutions exist ($x_i > 0$ and $p_i > 0$ in the symmetric equilibrium). If $p_i = 0$ then profits are zero and the firm would find increasing p_i to be profitable. To analyze $x_i = 0$, we have to specify what happens with s_i when $x_i = 0$. Assume that if $x_i = 0$ and $x_j > 0$ for at least one $j \neq i$ then $s_i = 0$. When $x_i = 0$ for all i, on the other hand, $s_i = \frac{\exp(-\delta\, p_i)}{\sum \exp(-\delta\, p_i)}$. Three cases exist: $x_p = 0$ and $x_o > 0$, $x_p > 0$ and $x_o = 0$, and $x_p = 0$ and $x_o = 0$. If $x_p = 0$ and $x_o > 0$ then $s_p = 0$ and a P firm makes zero profits. But a P firm can deviate to $p_i = \frac{1}{\delta\,(1-s_i)}$ and $x_i = \frac{s_i}{c}$ with $s_i > 0$. Such a deviation is profitable if $s_i > 1 - \frac{1}{\delta}$, which always holds. If $x_p > 0$ and $x_o = 0$ then $s_o = 0$ and an O firm makes zero profits. But an O firm can deviate to $p_i = \frac{1}{\delta(1-s_i)}$ and $x_i = \frac{s_i}{c}$ obtaining positive profits. If $x_p = 0$ and $x_o = 0$ then $s_p = s_o = \frac{1}{n}$. An O or a P firm can deviate to $x_i = \epsilon > 0$ obtaining a discontinuous jump in revenue ($s_i = 1$) and a small increase in costs.

Finally, to show existence and uniqueness, we need to prove two things: (1) only one fixed point of the system of equations in Proposition 1 exists (only one symmetric equilibrium exists), and (2) the profit function is concave at the equilibrium (the second-order conditions for optimality hold).

Let us first show only one fixed point exists in terms of equilibrium market shares. Define the function $g(s_o)$ by plugging equation (7) in equation (6):

$$g(s_o) = (1-\delta)\ln\left(\frac{(n-n_o)s_o}{1-n_o s_o}\right) - \delta\,\ln\left(1-\alpha\frac{n_o-1}{(1-s_o)n_o}\right) + \tag{25}$$
$$-\,\alpha\,\delta\,\ln(n_o) - \frac{n-n_o}{n-1-n_o(1-s_o)} + \frac{1}{1-s_o}.$$

By construction, s_o solves equations (6) and (7) if and only if $g(s_o)=0$. Existence of at least one s_o such that $g(s_o)=0$ follows from continuity of g and the fact that $\lim_{s_o\to 0} g(s_o) = -\infty$ and $\lim_{s_o\to\frac{1}{n_o}} g(s_o) = +\infty$. To show that only one such s_o exists, it is sufficient to show that g is strictly increasing. Computing this derivative, we obtain

$$\frac{\partial g}{\partial s_o} = \frac{1-\delta}{s_o(1-n_o\,s_o)} + \frac{\alpha\,\delta\,(n_o-1)/(1-s_o)}{(1-\alpha)(n_o-1)+1-s_o\,n_o} +$$
$$+\,\frac{(n-n_o)\,n_o}{(1+n_o(1-s_o)-n)^2} + \frac{1}{(1-s_o)^2}.$$

All terms are positive because $s_o n_o \leq 1$. It follows that a unique (s_o, s_p) exists solving the system of equations.

To prove the profit function is concave at the equilibrium candidate, we will show the Hessian of the profit function (at the equilibrium price and market share) is negative definite. A necessary and sufficient condition for negative definiteness is that the leading principal minors alternate sign. To simplify the exposition, we only show the determinants of the Hessian of both firms:

$$|H_p| = \frac{\delta\,s_p^2}{x_p^2}\left(1-\delta\,(1-s_p)^2\right),$$
$$|H_o| \geq \delta\frac{s_o^2}{x_o^2}\left(\left(\frac{1-n_o\,s_o}{(1-s_o)n_o^2}\alpha + (1-\alpha)\right) - \delta\,(1-s_o)^2\left(\frac{1-n_o\,s_o}{(1-s_o)n_o}\alpha + (1-\alpha)\right)^2\right).$$

A sufficient condition for both determinants to be positive is $\delta \leq 1$, which we have assumed throughout the paper. Thus the concavity of the profit function at the equilibrium is guaranteed for both types of firms. ■

Proof of Lemma 1. To prove the first part of the lemma, we only have to check the sign of $g\left(\frac{1}{n}\right)$, where g is defined in (25). If $g\left(\frac{1}{n}\right) < 0$ then $s_o > 1/n$; therefore, $s_o > s_p$. Then

$$g\left(\frac{1}{n}\right) = -\delta\left(\ln\left(1-\alpha\frac{n}{n-1}\frac{n_o-1}{n_o}\right) + \alpha\ln(n_o)\right),$$

and $g\left(\frac{1}{n}\right) < 0$ if and only if

$$\alpha\,\frac{n_o^\alpha}{n_o^\alpha-1}\,\frac{n_o-1}{n_o} > \frac{n-1}{n}.$$

Let $h(\alpha, n_o) = \alpha\,\frac{n_o^\alpha}{n_o^\alpha-1}\,\frac{n_o-1}{n_o}$. Given that $h(\alpha, n_o)$ is increasing in α, and $h(0, n_o) < (n-1)/n < h(1, n_o)$, only one $\hat{\alpha}$ exists such that $h(\hat{\alpha}, n_o) = (n-1)/n$. Moreover, for $\alpha > \hat{\alpha}$, we have $h(\alpha, n_o) > (n-1)/n$ and $s_p > s_o$, and viceversa.

Finally, the proof that $\hat{\alpha}(n_o, n)$ is increasing in n and n_o follows from applying the implicit function theorem to $F(\alpha, n_o, n) = h(\alpha, n_o) - (n-1)/n$, and by observing that $\partial h/\partial n_o < 0$. ■

Proof of Lemma 2. By the implicit function theorem, $\frac{\partial s_o}{\partial \alpha} = -\frac{\partial g/\partial \alpha}{\partial g/\partial s_o}$. We know $\partial g/\partial s_o > 0$. Let us now compute $\partial g/\partial \alpha$:

$$\frac{\partial g}{\partial \alpha} = \ln(n_o) - \frac{n_o - 1}{\alpha + (1 - s_o - \alpha)n_o}.$$

Therefore, $\partial s_o/\partial \alpha = 0$ when $\partial g/\partial \alpha = 0$. Solving for the value $\hat{s}_o$ that makes $\partial g/\partial \alpha = 0$, we obtain

$$\hat{s}_o = \frac{\ln(n_o)(n_o(1-\alpha)+\alpha) + 1 - n_o}{n_o \ln(n_o)}.$$

Substituting $\hat{s}_o$ in $g = 0$, we obtain an equation determining the value α_s that makes the derivative equal to zero. To prove that to the right of α_s the graph of $s_o(\alpha)$ is decreasing, assume to the contrary $\partial g/\partial \alpha > 0$. Then for $\alpha > \alpha_s$, it has to be the case that $s_o > \hat{s}_o$ which implies $\partial g/\partial \alpha < 0$, a contradiction. Then $\partial s_o/\partial \alpha < 0$ for $\alpha > \alpha_s$. A similar reasoning implies $\partial s_o/\partial \alpha > 0$ for $\alpha < \alpha_s$. ∎

Proof of Lemma 3. By the implicit function theorem, $\frac{\partial s_o}{\partial \delta} = -\frac{\partial g/\partial \delta}{\partial g/\partial s_o}$. In the proof of Proposition 1, we showed $\partial g/\partial s_o > 0$. Next we will determine the sign of $\partial g/\partial \delta$. Computing this derivative, we obtain

$$\begin{aligned}\frac{\partial g}{\partial \delta} &= -\alpha \ln n_o - \ln\left(1 - \alpha\frac{n_o - 1}{n_o(1-s_o)}\right) - \ln\left(\frac{(n-n_o)s_o}{1-n_o s_o}\right) \\ &= -\frac{1}{\delta}\left[(1-\delta)\ln\left(\frac{s_o}{s_p}\right) + \frac{1}{1-s_o} - \frac{1}{1-s_p}\right] - \ln\left(\frac{(n-n_o)s_o}{1-n_o s_o}\right). \qquad (26)\end{aligned}$$

In the second row, we use the expression for $g(s_o) = 0$. If $\alpha > \hat{\alpha}$ then $s_o < s_p$ (by Lemma 1), $\partial g/\partial \delta > 0$ (by equation (26)), and $\partial s_o/\partial \delta < 0$. Conversely, if $\alpha < \hat{\alpha}$ then $s_o > s_p$, $\partial g/\partial \delta < 0$, and $\partial s_o/\partial \delta > 0$. ∎

Proof of Proposition 2. We begin by showing existence. For $n_o = 1$ to be an equilibrium, we only need $f(2) \leq 0$. Likewise, for $n_o = n$ to be an equilibrium, we only need $f(n) \geq 0$. To have an equilibrium with both types of firms ($1 < n_o < n$), we need $f(n_o) \geq 0$ and $f(n_o{+}1) \leq 0$ at the equilibrium n_o. Suppose no equilibrium exists with $n_o = 1$ or $n_o = n$. Then $f(2) > 0$ and $f(n) < 0$, so $f(n_o)$ goes from positive to negative at least once when going from $n_o = 1$ to $n_o = n$. Therefore, existence of an equilibrium is guaranteed.

Next, we show $f(2) > 0$ for any n, α, and δ, which means the equilibrium always has at least two O firms. From the definition of $f(n_o)$, we obtain

$$\delta\, f(2) = \frac{s_o(2)}{1 - s_o(2)}\left(1 - \delta\,(1 - s_o(2)) + \frac{\alpha\delta}{2}\right) - \frac{1}{n-1}\left(1 - \delta\,\frac{n-1}{n}\right).$$

Let $\bar{s}_o$ be the maximum value of $s_o(2)$ for which $f(2) \leq 0$. From the above equation, we obtain

$$\bar{s}_o = \frac{\alpha\, n - 2}{4\,(n-1)\,n} - \frac{n\left(1 - \delta\left(1 - \frac{\alpha}{2}\right)\right)}{2\,\delta\,(n-1)} + \sqrt{\frac{\delta + (1-\delta)\,n}{\delta\,(n-1)\,n} + \left(\frac{n}{2\,\delta\,(n-1)} - \frac{\left(1 + \left(1 - \frac{\alpha}{2}\right)n\right)}{2\,n}\right)^2}.$$

Let $w = g(\bar{s}_o)$, where $g(s_o)$ is defined in (25). $w < 0$ implies $s_o(2) > \bar{s}_o$, which means $f(2) > 0$. w is strictly increasing in n and has the following upper bound as $n \to \infty$:

$$\bar{w} = -\alpha\,\delta\,\ln(2) + (1-\delta)\,\ln\left(\frac{1-\delta}{1 - \delta\left(1 - \frac{\alpha}{2}\right)}\right) - \delta \ln\left(1 - \frac{\alpha}{2}\right).$$

$\bar{w}$ is strictly convex in δ and α, which means the maximum is at $\delta = 0$ or $\delta = 1$, and $\alpha = 0$ or $\alpha = 1$. It is straightforward to show $\bar{w}$ goes to zero as δ or α go to zero, and also when both δ and α go to 1. Given that w is strictly increasing in n, we conclude that $w(n, \delta, \alpha)$ is negative for any finite n. Therefore, $f(2) > 0$.

Lemma A1 will prove important in characterizing the subgame perfect equilibrium of the game. For an O firm to find it profitable to become P ($f(n_o) < 0$), the increase in market share from becoming P must be large enough to compensate for the increase in cost. If $\alpha < \hat{\alpha}(n_o{-}1, n)$, O firms have a larger market share, so deviating is not profitable to them ($f(n_o) > 0$). Corollaries A1 and A2 are two important implications of this lemma.

Lemma A1 (Sufficient condition for positive f)**.** *If $\alpha < \hat{\alpha}(n_o{-}1, n)$ then $f(n_o) > 0$.*

Proof. Rearranging $f(n_o)$ and multiplying by δ, we obtain

$$\delta\, f(n_o) = \frac{s_o}{1 - s_o}(1 - \delta(1 - s_o)) - \frac{\tilde{s}_p}{1 - \tilde{s}_p}(1 - \delta(1 - \tilde{s}_p)) + \alpha\delta\frac{s_o}{1 - s_o}\frac{n_o{-}1}{n_o},$$

where $s_o = s_o(n_o)$ and $\tilde{s}_p = s_p(n_o - 1)$. The sign of f depends on the sign of the right-hand side of the equation. The first two terms have the same functional form and are increasing in s. The last term is always positive. Therefore, if $s_o(n_o) \geq s_p(n_o{-}1)$, then $f(n_o) > 0$. A sufficient condition is that $s_o(n_o{-}1) \geq 1/n$ and $s_o(n_o) \geq 1/n$, which is equivalent to $\alpha < \hat{\alpha}(n_o{-}1, n)$ and $\alpha < \hat{\alpha}(n_o, n)$. However, $\hat{\alpha}(n_o, n)$ is decreasing in n_o, so $\alpha < \hat{\alpha}(n_o{-}1, n)$ implies $f(n_o) > 0$. ■

Corollary A1 (Necessary condition for an interior equilibrium)**.** *At an interior equilibrium n_o, it is necessary that $\alpha \geq \hat{\alpha}(n_o, n)$.*

Proof. For an interior equilibrium at n_o, it is necessary $f(n_o) \geq 0$ and $f(n_o{+}1) \leq 0$, but Lemma A1 implies that for $f(n_o{+}1) \leq 0$, we need $\alpha \geq \hat{\alpha}(n_o{-}1, n)$. ■

Corollary A2 (Sufficient condition for an equilibrium with $n_o = n$)**.** *If $\alpha \leq \hat{\alpha}(n - 1, n)$, all firms decide to be O in equilibrium.*

Proof. If $\alpha \leq \hat{\alpha}(n - 1, n)$ then $f(n) \geq 0$, so if $n_o = n$, no firm would gain by becoming a P firm. ■

Corollary A1 states that in any interior equilibrium, the P firms must have a larger market share than O firms; therefore, a higher quality product. Corollary A2, on the other hand, shows that if the degree of public good of the investment is low enough, O firms have a larger market share for any n_o; therefore, all firms decide to collaborate in the open-source project. Lemma A2 complements Corollary A2, by providing the necessary and sufficient condition for an equilibrium with $n_o = n$.

Lemma A2 (Necessary and Sufficient condition for equilibrium with $n_o = n$)**.** *Given $n > 3$ and δ, $\bar{\alpha} \in (\hat{\alpha}, 1)$ exists such that $f(n) \geq 0$ if and only if $\alpha \leq \bar{\alpha}$.*

Proof. The equilibrium condition with $n_o = n$ depends only on the sign of $f(n_o)$, so for simplification, we will work with a scaled version of $f(n_o)$, $\delta\, f(n_o)$, for the rest of this proof. We know $f(n) > 0$ for $\alpha < \hat{\alpha}(n - 1, n)$. We need to determine the sign of $f(n)$ for the rest of values of α. When $n_o = n$, $s_o = 1/n$. Therefore,

$$f(n) = \frac{1}{n-1} - \frac{\delta(1-\alpha)}{n} - \frac{\tilde{s}_p}{1 - \tilde{s}_p}\left(1 - \delta(1 - \tilde{s}_p)\right), \tag{27}$$

where $\tilde{s}_p = s_p(n-1)$. We need to find the value of $\tilde{s}_p$ that makes $f(n) = 0$. This equation has two roots. The only positive root is

$$\tilde{s}_p = \frac{-n^2(1-\delta) - (1-\alpha)\delta - n\alpha\delta + \sqrt{n^4 - 2(n-1)n^2(n-1-\alpha)\delta + z^2}}{2\delta\, n\,(n-1)},$$

where $z = \delta\,(n-1)\,(n-1+\alpha)$. The corresponding value for $s_o(n-1)$ is

$$\tilde{s}_o = \frac{n^2 + z - \sqrt{n^4 - 2(n-1)n^2(n-1-\alpha)\delta + z^2}}{2\delta\, n\,(n-1)^2}.$$

Plugging this value in the equilibrium condition (25) and solving for α, we obtain the value $\bar{\alpha}$, where $f(n) = 0$. Lemma A1 implies $\bar{\alpha} \geq \hat{\alpha}(n-1, n)$. Lemma 2 implies $\partial\tilde{s}_p/\partial\alpha > 0$ in the relevant area. Then $\bar{\alpha}$ is the unique value of α such that $f(n) = 0$.

To finish the proof, we need to show $f(n) > 0$ for $\alpha < \bar{\alpha}$ and $f(n) < 0$ for $\alpha > \bar{\alpha}$. Given the continuity and monotonicity of s_p, it suffices to show some value to the right or to the left of $\bar{\alpha}$ exists such that these inequalities hold.

Consider first the case of $\alpha < \bar{\alpha}$. We know that at $\alpha = \hat{\alpha}(n-1, n)$, $f(n) > 0$. Then $f(n) > 0$ for $\alpha < \bar{\alpha}$. For $\alpha > \bar{\alpha}$, consider $\alpha = 1$. When $\alpha = 1$, the investment of O firms is low, and P firms have the largest advantage. In this case, $f(n) < 0$, which proves this inequality holds for any $\alpha > \bar{\alpha}$. ■

Proposition 2 follows directly from Corollaries A1 and A2, and Lemma A2. ■

Proof of Proposition 3. When $n \to \infty$, three types of equilibria exist. Remember we defined $\pi_k(n_o)$ as the profit of firm type k when there are n_o O firms. For an equilibrium with coexistence, we need $\lim_{n\to\infty} \frac{\pi_o(n_o)}{\pi_p(n_o)} = 1$. For an equilibrium with only O firms, we need $\lim_{n\to\infty} \frac{\pi_o(n)}{\pi_p(n)} \geq 1$, and for an equilibrium with only P firms, we need $\lim_{n\to\infty} \frac{\pi_o(2)}{\pi_p(1)} \leq 1$.

Taking the limit of equation 6, we obtain a function determining the limit of the ratio of market shares,

$$(1-\delta)\ln \lim_{n\to\infty} \frac{s_o}{s_p} = \delta \ln \lim_{n\to\infty} \left[\left(1 - \alpha\frac{n_o - 1}{n_o(1-s_o)}\right) n_o^{\alpha}\right],$$

and from the definition of π_o and π_p in equations (8) and (9), we obtain the ratio of profits,

$$\lim_{n\to\infty} \frac{\pi_o}{\pi_p} = \lim_{n\to\infty} \frac{1 - \delta\left(1 - \alpha\frac{n_o-1}{n_o}\right)}{1-\delta} \lim_{n\to\infty} \frac{s_o}{s_p}.$$

In an equilibrium with coexistence, $n_o \to \infty$ as $n \to \infty$. Two cases exist. If $\alpha < 1$ then $\frac{s_o}{s_p} \to \infty$ and $\frac{\pi_o}{\pi_p} \to \infty$, which contradicts the necessary condition for an equilibrium with coexistence. $\alpha = 1$, on the other hand,

$$(1-\delta)\ln \lim_{n\to\infty} \frac{s_o}{s_p} = \delta \ln \lim_{n_o\to\infty} (1 - n_o\, s_o),$$

so $\frac{s_o}{s_p}$ may converge to a constant if $n_o\, s_o$ converges to a constant. Specifically, if $n_o\, s_o \to 1 - (1-\delta)^{\frac{1-\delta}{\delta}}$, then $\frac{s_o}{s_p} \to 1 - \delta$ and $\frac{\pi_o}{\pi_p} \to 1$, and an equilibrium with coexistence exists.

Let us now look for an equilibrium with only O firms. If $\alpha < 1$, $\frac{s_o}{s_p} \to \infty$ and $\frac{\pi_o}{\pi_p} \to \infty$, which verifies the condition for an equilibrium with only O firms. If $\alpha = 1$, $n_o\, s_o \to 1$, which means $\frac{s_o}{s_p} \to 0$ and $\frac{\pi_o}{\pi_p} \to 0$, which contradicts the necessary condition for an equilibrium with only O firms.

Finally, let us look for an equilibrium with only P firms. We have to show that when $n_o = 2$, O firms would gain by becoming P. When $n_o = 2$,

$$\lim_{n\to\infty} \frac{s_o}{s_p} = \left[\left(1-\frac{\alpha}{2}\right)2^{\alpha}\right]^{\frac{\delta}{1-\delta}},$$
$$\lim_{n\to\infty} \frac{\pi_o(2)}{\pi_p(1)} = \frac{1-\delta\,\left(1-\alpha\frac{1}{2}\right)}{1-\delta}\left[\left(1-\frac{\alpha}{2}\right)2^{\alpha}\right]^{\frac{\delta}{1-\delta}}.$$

The above expression is larger than 1 for all α and δ. Therefore, no equilibrium with only P firms exists. ■

Proof of Lemma 4. The equilibrium condition for the second stage is fully characterized by

$$(1-\delta)\,\ln\left(\frac{s_o}{s_p}\right)+\frac{1}{1-s_o}-\frac{1}{1-s_p}=\delta\,\ln\left(1-\alpha\,\frac{n_o-1}{n_o(1-s_o)}\right)+\alpha\,\delta\,\ln\left(n_o\right)-\delta\ln\left(1-\kappa\right), \quad (28)$$

and the market clearing condition $n_o\,s_o+(n-n_o)\,s_p=1$. The left-hand side in equation 28 is zero when $s_o = s_p$ and it is strictly increasing in s_o. Then $s_o > s_p$ if and only if the right-hand side is positive. Finally, the right-hand side is positive if and only if $1-\kappa < \left(1-\alpha\frac{n}{n-1}\frac{n_o-1}{n_o}\right)n_o^{\alpha}$. ■

Proof of Proposition 4. Suppose the subsidy is high enough such that all firms decide to be O. Individual investments are $x_o = (1-\alpha)/(n\,c_o)$. The government wants to find the subsidy that attains the optimal investment $x^* = 1/(c\,n)$. Thus the optimal subsidy is $\kappa^* = \alpha$. Now we have to show that given subsidy κ^*, all firms choose to be O. Remember that if $\kappa > 1-\left(1-\alpha n(n-2)(n-1)^{-2}\right)(n-1)^{\alpha}$ then $s_o > s_p$ for $n_o = n-1$; therefore, all firms want to be O. Given that $n(n-2)(n-1)^{-2}\in[0,1]$, $\kappa > 1-(1-\alpha)(n-1)^{\alpha}$ is a sufficient condition, which holds if $\kappa = \alpha$. ■

Proof of Proposition 5. The first-order conditions are (16) and (17). As in the standard logit model, no corner solutions exist so the first-order conditions hold with equality. Equilibrium prices and investment for P firms are identical to the logit model, so we will focus on the O firms. In the case of O firms, the partial derivative of the market share with respect to the price is

$$\frac{\partial s_i}{\partial p_i} = -\frac{\delta}{(1-\sigma)}s_i(1-\sigma\,s_{i|os}-(1-\sigma)\,s_i).$$

Then from the equation (16) and imposing symmetry, we obtain the optimal price in equation (13). To find x_o, we need to calculate $\partial s_i/\partial x_i$ for O firms:

$$\frac{\partial s_i}{\partial x_i} = \delta\,(1-\sum_{i\in O}s_i)\frac{\alpha}{\sum_{i\in O}x_i}+\delta\frac{s_i(1-\sigma\,s_{i|os}-(1-\sigma)\,s_i)}{(1-\sigma)}\frac{(1-\alpha)}{x_i}.$$

From equation (17) and imposing symmetry, we obtain equation (14), and the ratio of optimal investments in equilibrium:

$$\frac{x_o}{x_p} = \frac{s_o}{s_p}\left(1-\frac{\alpha}{1-\sigma}\,\frac{(n_o-1)/n_o}{\left(1-s_o+\frac{\sigma}{1-\sigma}\frac{n_o-1}{n_o}\right)}\right). \quad (29)$$

The ratio of market shares between O and P firms is

$$\frac{s_o}{s_p} = n_o^{-\sigma} \exp\left(\delta(q_o - q_p + p_p - p_o)\right), \tag{30}$$

$$\ln\left(\frac{s_o}{s_p}\right) = -\sigma \ln n_o + \delta\left(q_o - q_p\right) + \frac{1}{1-s_p} - \frac{1}{1 - s_o + \frac{\sigma}{1-\sigma}\frac{n_o-1}{n_o}}, \tag{31}$$

and from the definition of q_p and q_o, we obtain:

$$q_o - q_p = \ln\left(\frac{x_o}{x_p}\right) + \alpha \ln\left(n_o\right). \tag{32}$$

From equations (29), (31), and (32), we obtain equation (15), which is an implicit equation determining the relation of market shares between O and P firms in equilibrium. This equation, together with the equation establishing the sum of the market shares is equal to 1, completely characterizes the equilibrium.

Finally, to show existence and uniqueness, we need to prove two things: (1) only one fixed point of the system of equations in Proposition 5 exist (only one symmetric equilibrium exists), and (2) the profit function is concave at the equilibrium (the second-order conditions for optimality hold).

To show (1), define $g(s_o)$ by plugging (7) in equation (15). Then the result follows from an application of the mean value theorem as in the standard logit model.

To prove the profit function is concave at the equilibrium candidate, we will evaluate the determinant of the Hessian of the profit function at the equilibrium price and market share, and show it is positive definite. The determinant of the Hessian for O firms is

$$\begin{aligned}|H_o| \geq \delta \frac{s_o^2}{(1-\sigma)\,x_o^2}(1-\sigma s_{i|os}) \Bigg(\frac{(1-\sigma)}{\delta}\left(\frac{(1-\sigma)(1-n_o\, s_o)}{(1-\sigma s_{i|os}-(1-\sigma)s_o)n_o^2}\alpha + (1-\alpha)\right) + \\ - \left(\frac{(1-\sigma)(1-n_o\, s_o)}{(1-\sigma s_{i|os}-(1-\sigma)s_o)n_o}\alpha + (1-\alpha)\right)^2\Bigg).\end{aligned}$$

The determinant of the Hessian for P firms is equivalent to that of the standard logit model. A sufficient condition for both determinants to be positive is $(1-\sigma) \geq \delta$ or $\sigma \leq (1-\delta)$, which we have assumed for this section of the paper. Thus, the concavity of the profit function at the equilibrium is guaranteed for both types of firms. ■

Proof of Proposition 6. We have characterized the equilibrium when $\sigma = 0$ in Proposition 3. Therefore, for the rest of the proof, assume $\sigma > 0$. Taking the limit of equation (15), we obtain the ratio of market shares in equilibrium, and from the definition of π_o and π_p, we obtain the ratio of profits,

$$\lim_{n\to\infty} \frac{\pi_o}{\pi_p} = \lim_{n\to\infty} \frac{1}{1 + \frac{\sigma}{1-\sigma}\frac{n_o-1}{n_o}} \lim_{n\to\infty} \frac{1-\delta+\frac{\delta}{1-\sigma}\left(\alpha-\sigma\right)\frac{n_o-1}{n_o}}{1-\delta} \lim_{n\to\infty}\frac{s_o}{s_p}.$$

First, we look for equilibria with $n_o \to \infty$ (coexistence or only O firms). If $\alpha < 1$, then

$$(1-\delta)\ln \lim_{n\to\infty} \frac{s_o}{s_p} = \sigma + \delta \ln\left[(1-\alpha) \lim_{n_o\to\infty} n_o^{\alpha-(\sigma/\delta)}\right].$$

$\alpha < 1$ in three cases. If $\alpha > \sigma/\delta$ then $\frac{s_o}{s_p} \to \infty$ and $\frac{\pi_o}{\pi_p} \to \infty$ when $n_o \to \infty$; therefore, an equilibrium with only O firms exists. If $\alpha < \sigma/\delta$ then $\frac{s_o}{s_p} \to 0$ and $\frac{\pi_o}{\pi_p} \to 0$; therefore, no equilibrium exists in which $n_o \to \infty$. Finally, if $\alpha = \sigma/\delta$ then $\frac{s_o}{s_p} \to e^{\frac{\sigma}{1-\delta}} \left(1 - \frac{\sigma}{\delta}\right)^{\frac{\delta}{1-\delta}} < 1$ and $\frac{\pi_o}{\pi_p} \to \frac{s_o}{s_p}$; therefore, no equilibrium exists in which $n_o \to \infty$. If $\alpha = 1$ then

$$(1-\delta) \ln \lim_{n\to\infty} \frac{s_o}{s_p} = \sigma + \delta \ln \lim_{n_o\to\infty} \left[(1-\sigma)\,(1 - n_o\, s_o)\, n_o^{-\sigma/\delta}\right],$$

so $\frac{s_o}{s_p} \to 0$ and $\frac{\pi_o}{\pi_p} \to 0$ when $n_o \to \infty$; therefore, no equilibrium exists in which $n_o \to \infty$. Therefore, an equilibrium exists with $n_o \to \infty$ only when $\sigma/\delta < \alpha < 1$, and in such equilibrium, all firms choose to be O.

Next we look for an equilibrium with only P firms, for which we need $\lim_{n\to\infty} \frac{\pi_o(2)}{\pi_p(1)} \leq 1$. Given that firms are infinitesimal when $n \to \infty$, the market share of P firms is the same when $n_o = 1$ and $n_o = 2$. Taking the limit of equation (15), we obtain

$$\lim_{n\to\infty} \frac{s_o}{s_p} = 2^{\frac{\alpha\,\delta - \sigma}{1-\delta}}\; e^{\frac{\sigma}{(1-\delta)(2-\sigma)}} \left(1 - \frac{\alpha}{2-\sigma}\right)^{\frac{\delta}{1-\delta}};$$

therefore, the limit of the ratio of profits is

$$\lim_{n\to\infty} \frac{\pi_o(2)}{\pi_p(1)} = \frac{2\,(1-\sigma)\,(1-\delta) + \delta\,(\alpha - \sigma)}{(1-\delta)\,(2-\sigma)}\, 2^{\frac{\alpha\,\delta-\sigma}{1-\delta}}\; e^{\frac{\sigma}{(1-\delta)(2-\sigma)}} \left(1 - \frac{\alpha}{2-\sigma}\right)^{\frac{\delta}{1-\delta}}.$$

Denote the right-hand side of the above expression by $F(\alpha, \delta, \sigma)$, and let $\tilde{\sigma}(\alpha, \delta)$ be the value of σ that solves $F(\alpha, \delta, \sigma) = 1$. $F(\alpha, \delta, \sigma)$ is decreasing in σ, which means $\lim_{n\to\infty} \frac{\pi_o(2)}{\pi_p(1)} \leq 1$ (i.e., an equilibrium with only P firms exists) if and only if $\sigma \geq \tilde{\sigma}(\alpha, \delta)$. Note this condition holds not only for $\alpha < 1$, but also for $\alpha = 1$.

Next we show $0 < \tilde{\sigma} < \alpha\,\delta$. We know $\tilde{\sigma} > 0$ from Proposition 3. Also, $\tilde{\sigma} \geq \alpha\,\delta$ if and only if $F(\alpha, \delta, \alpha\,\delta) \geq 1$. Substituting $\sigma = \alpha\,\delta$ into $F(\alpha, \delta, \sigma)$ and rearranging, we obtain the following condition:

$$\frac{\alpha}{2 - \alpha\,\delta} + \ln\left(1 - \frac{\alpha}{2 - \alpha\,\delta}\right) \geq 0,$$

which is not possible, given that $\alpha/(2 - \alpha\,\delta) > 0$. Therefore, $\tilde{\sigma} < \alpha\,\delta$.

Finally, we know that when $\alpha = 1$, no equilibrium exists such that $n_o \to \infty$; that is, no equilibrium exists in which all firms choose to be O, and no equilibrium exists in which a proportion of firms chooses to be O. We also know an equilibrium with only P firms exists only when $\sigma \geq \tilde{\sigma}(\alpha, \delta)$. Therefore, it remains to show what the equilibrium is when $\alpha = 1$ and $0 < \sigma < \tilde{\sigma}(\alpha, \delta)$. We can easily see that in this case, an equilibrium with coexistence exists, in which the number of O firms converges to a constant. To see this, note that $\sigma < \tilde{\sigma}(\alpha, \delta)$ implies $\frac{\pi_o(2)}{\pi_p(1)} > 1$, and that we have shown $\lim_{n_o\to\infty} \frac{\pi_o}{\pi_p} < 1$ when $\alpha = 1$. Therefore, a constant $2 \leq n_o^* < \infty$ exists such that $\frac{\pi_o(n_o^*)}{\pi_p} \geq 1$ and $\frac{\pi_o(n_o^*+1)}{\pi_p} \leq 1$ (the market share of P firms does not change when n_o changes and n_o is finite, because in this case, O firms have an aggregate market share equal to zero). ■

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